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Modifying Minkowski's Theorem

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At the turn of the century, Minkowski published his famous "convex body" theorem which became the basis for the geometry of numbers. Suppose that Λ is a lattice in Euclidean n-space, E^n, having determinant $d(\Lambda)$. Now Minkowski's theorem states that if K is a convex body which is symmetric about the origin O, and if K contains no nonzero points of the lattice Λ, then the volume $V(K)$ of K satisfies $V(K) \leqslant 2^n d(\Lambda)$. In spite of its simple nature, Minkowski's theorem is a powerful and important result. Since he theorem first appeared, there have been a surprisingly large number of modifications and variations published. A number of these will be discussed, particularly some of the more recent discoveries and unproved conjectures.

1. INTRODUCTION

Let Λ be a lattice in n-dimensional space E^n, having determinant $d(\Lambda)$. We have the classical theorem of Minkowski (see [1911]):

THEOREM. *Let $K \subset E^n$ be an open point set of volume $V(K)$ which satisfies*

(a) *The Convexity Condition: K is convex.*

(b) *The Symmetry Condition: K is symmetric about the origin O (O-symmetric).*

(c) *The Lattice Point Condition: K contains no non-zero point of Λ.*

Then

$$V(K) \leqslant \mu_n d(\Lambda) \tag{1}$$

where $\mu_n = 2^n$.

Minkowski's work gave rise to the *geometry of numbers*, a link between number theory and geometry, and much work has been done in this area

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(see, for example, Cassels [1959] or Lekkerkerker [1969]). If we take Λ to be the integer lattice Z^n, and K the cube, $|x_i| \leqslant 1$ $(i=1, ..., n)$, we see that the constant 2^n cannot be improved. On the other hand, if K is a given body (for example, a sphere or a cylinder), we can set $\Delta(K) = \inf d(\Lambda)$, taken over all lattices Λ satisfying condition (c) of the theorem, and $V(K)/\Delta(K)$ is a suitable constant for the given K. We notice that set K need not in fact be convex or symmetric for the lattice constant $\Delta(K)$ to be determined.

Let now K satisfy conditions (a) and (b) of the theorem, and consider λK $(\lambda \geqslant 0)$. As λ increases, there will be a least λ_1 for which $\lambda_1 K$ first has a non-zero lattice point on its boundary, and (1) takes the form

$$\lambda_1^n V(K) \leqslant 2^n d(\Lambda). \tag{2}$$

Allowing λ to increase further, and defining

$$\lambda_i = \inf\{\lambda \mid \lambda \geqslant 0, \dim(\lambda K \cap \Lambda) \geqslant i\} \qquad (i=1, ..., n),$$

we obtain the so-called *successive minima* $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ of K. Minkowski obtained the stronger result

$$\lambda_1 \lambda_2 \cdots \lambda_n V(K) \leqslant 2^n d(\Lambda). \tag{3}$$

Simpler proofs have been given by Davenport [1939], Weyl [1942], Bambah, Woods, and Zassenhaus [1965], and Danicic [1969]. Clearly (3) implies (2).

Finally we might mention that Minkowski investigated the sets K for which equality holds in (1). He determined that K must be a polytope with not more than $2(2^n - 1)$ faces, and that there are at most $3^n - 1$ lattice points on the boundary of K. Swinnerton-Dyer [1953] established the nice lower bound of $n(n+1)$ lattice points on the boundary of K for a "critical" lattice Λ. Van der Corput and Davenport [1946] have shown that any set K satisfying the conditions of Theorem 1 is contained in a polytope Π which also satisfies the conditions. Wills [1982] has investigated the placing of the boundary points in the case when K is a lattice polytope.

2. The Convexity Condition

It is clear that we can expect no bound on the volume $V(K)$ if the convexity condition on K is completely omitted. However, the condition can be slackened to yield some interesting analogues of Minkowski's theorem.

It is easily seen that, subject to suitable continuity conditions, K is convex if and only if

$$x,\ y \in K \Rightarrow \frac{x+y}{2} \in K.$$

The number 2 in the denominator appears to be significant for Minkowski's theorem. If, following Mordell [1934], we replace this condition by

$$x,\ y \in K \Rightarrow \frac{x+y}{k} \in K, \tag{4}$$

then for fixed $k > 2$ we introduce a measure of non-convexity. Mordell shows that if K is O-symmetric, satisfies (4), and contains no non-zero lattice points, then

$$V(K) \leqslant k^n\, d(\Lambda)..$$

The idea was elaborated by van der Corput [1936], and generalized by Rado [1946]. Rado combined the "convexity" and symmetry conditions by replacing (4) by

$$x,\ y \in K \Rightarrow A(x-y) \in K,$$

where A is an $n \times n$ matrix. Convex, O-symmetric sets correspond to $A = \frac{1}{2} I$. The analytical inequality obtained by Rado has since been strengthened by Cassels [1947] and Uhrin [1980, 1981].

We can also improve the constant in (1) by tightening the convexity condition. For $n = 2$, van der Corput and Davenport [1946] consider lattices with $d(\Lambda) = 1$, and replace the convexity condition by

> "the boundary of K has continuous radius of curvature ρ with $\rho \geqslant \rho_0 > 0$."

For such K, the area satisfies

$$A(K) \leqslant 4 - (2\sqrt{3} - \pi)\, \rho_0^2.$$

This bound is best possible, being obtained for a circle K and the regular hexagonal lattice. Jarnik [1948] derives a rather more fearsome bound by applying the same ideas to the stronger Minkowski inequality (3).

Both van der Corput and Davenport [1946] and Melzak [1959] investigate the general n-dimensional analogues. Melzak replaces the supporting hyperplane for a convex body K by a supporting n-sphere, which

641/29/1-2

supports K at a boundary point, and also contains K. I notice that each of these authors who have tightened the convexity condition have also retained the symmetry condition. We might ask whether this is in fact necessary.

Minkowski also noticed that for *strictly* convex sets K, there are at most $2^n - 1$ pairs of lattice points on the boundary. This result is put in the more general context of the successive minima by Woods [1958].

3. The Symmetry Condition

Most variations of Minkowski's theorem have been obtained by replacing or modifying the symmetry condition. Ehrhart led the way by replacing O-symmetry by the condition that the centre of gravity of K should lie at O. He conjectures [1964] that then in E^n, $V(K)/d(\Lambda) \leqslant (n+1)^n/n!$, and proves this in E^2 [1955a], and for solids of revolution in E^3 [1955b]. For each n, the (conjectured) bound is attained when K is a simplex [1964, 1979].

If we take Λ to be the integer lattice, we can exert some control over the shape of a convex set by specifying the side-length k of a smallest axis-oriented n-cube which contains K. We might then expect to obtain some function $\phi = \phi(k)$ such that $V(K) \leqslant \phi(k)$. Scott [174a, b] determines this function for $n = 2$; its graph is given in Fig. 1. A sample corresponding variation of a set K is illustrated in Fig. 2. We see that Ehrhart's critical triangle (Fig. 2b) corresponds to the local maximum in the graph of ϕ.

Many attempts have been made to constrain asymmetric sets K so as to keep the volume bounded. For example, we can insist on the circumcentre of K lying at the origin [Scott, 1982]. Or again, let O_i denote the ith

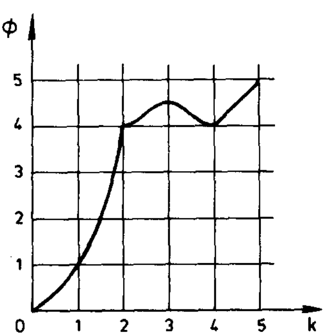

FIGURE 1

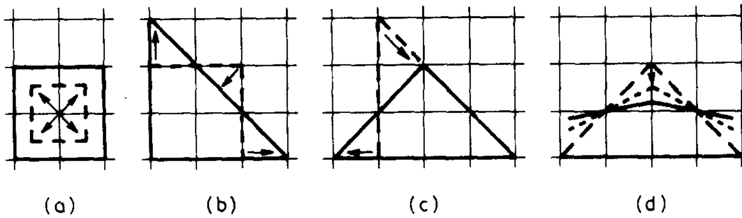

FIG. 2. (a) $0 \leqslant k \leqslant 2$, (b) $2 \leqslant k \leqslant 3$, (c) $3 \leqslant k \leqslant 4$, (d) $k \geqslant 4$.

orthant in E^n cut off by the coordinate hyperplanes. Then we have the conjecture [Scott, 1978], true for $n=2$, that

$$V(K \cap O_i) = \frac{1}{2^n} V(K) \ (1 \leqslant i \leqslant n) \Rightarrow V(K) \leqslant 2^n \, d(\Lambda).$$

It is also possible to introduce other measurements of the set K—diameter, perimeter, width, and so on. A discussion of these ideas would take us too far afield, but see, for example, Arkinstall and Scott [1979], Croft [1979], Hammer [1966], Reich [1970], Scott [1974a, b, 1975, 1982]. Wills, Zaks, and Perles [1982] have investigated the case where K is an asymmetric polytope.

A more direct approach is to introduce a *coefficient of asymmetry* λ for K. If POP' is an arbitrary chord of K through the origin O, then $\lambda = \lambda(K) = \sup PO/OP'$. Clearly $\lambda \geqslant 1$; equality occurs here when and only when K is O-symmetric. In three interesting papers [1954, 1955a, 1955b], Sawyer establishes the following results for asymmetric sets K.

(a) He shows that $V(K) \leqslant \gamma(\lambda, n) \, d(\Lambda)$, where

$$\gamma(\lambda, n) \leqslant (\lambda+1)^n \left\{ 1 - \left(\frac{\lambda-1}{\lambda} \right)^n \right\}.$$

(b) With some difficulty he obtains an exact formulation for $\gamma(\lambda, 2)$.

(c) He finds estimates for $\gamma(\lambda, n)$ for sets K which are symmetric in a point other than the origin.

The formulae in (b) are complicated, but the graph of $\gamma(\lambda, 2)$ bears an interesting (and perhaps not unexpected) resemblance to Fig. 1 (see Fig. 3). In (a), Sawyer strengthens an early result of Mahler [1939]. He obtains his result by applying Minkowski's theorem to a largest O-symmetric subset of $K \cup (-K)$.

Let a chord of K which is bisected by the origin O be called a *chord of symmetry* of K, and suppose that K has $s(K)$ such chords. For many years I felt it should be possible to replace the symmetry condition in Minkowski's

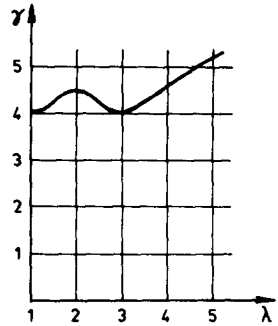

FIGURE 3

theorem by a far weaker condition. Arkinstall [1980a] shows that this is certainly the case in the plane. Thus he shows

$$s(K)=3 \qquad \Rightarrow A(K)\leqslant 4.5\, d(\Lambda)$$
$$s(K)=2 \text{ or } s(K)\geqslant 4 \Rightarrow A(K)\leqslant 4\, d(\Lambda).$$

The result continues to hold, for example, for chords of areal symmetry [1980b]. It is not clear how one might generalize this to E^n.

4. THE LATTICE POINT CONDITION

Minkowski's theorem can be extended in a nice way to sets which contain non-zero lattice points. Thus, following van der Corput [1936], replacing the lattice point condition by

> "K contains at most m distinct pairs of non-zero lattice points $\pm u_j$ $(1\leqslant j\leqslant m)$"

(as well as the origin), gives rise to

$$V(K)\leqslant (m+1)\, 2^n\, d(\Lambda).$$

Most of the previous variations can be combined with this new lattice condition; however, much less is known here. Since the flavour of Minkowski's theorem is only retained when the origin is considered as a special point, we restrict ourselves to these modifications.

Ehrhart [1955c, 1955d] has obtained incomplete results for planar convex sets with centre of gravity at the origin. Hammer [1966] has investigated O-symmetric sets satisfying an isoperimetric inequality. Perhaps the most promising results in the plane are due to Arkinstall. As before, let K possess $s(K)$ chords of symmetry through the origin, but now suppose that K contains c non-zero lattice points including the origin. Arkinstall shows [1982]

$$\text{If } s(K) \text{ is even or infinite, then } A(K)/d(\Lambda) \leqslant 2c+2.$$

$$\text{If } s(K) > 1 \text{ and } c \leqslant 4, \quad \text{then } A(k)/d(\Lambda) \leqslant 2c+2+1/(2c).$$

$$\text{If } s(K) > 3 \text{ and } c \leqslant 4, \quad \text{then } A(K)/d(\Lambda) \leqslant 2c+2.$$

The inequalities are best possible, but the proofs of the last two are long and involve much case-splitting. It seems likely that the restriction $c \leqslant 4$ is unnecessary. Perhaps this can best be shown by setting aside the symmetry condition, and proving that for "almost all" convex sets containing c lattice points, $A(K)/d(\Lambda) \leqslant 2c+2$.

References

J. R. Arkinstall, Minimal requirements for Minkowski's theorem in the plane I, *Bull. Austral. Math. Soc.* **22** (1980a), 259–274; Minimal requirements for Minkowski's theorem in plane II, *Bull. Austral. Math. Soc.* **22** (1980b), 274–284; "Generalizations of Minkowski's Theorem in the Plane," Doctoral thesis, University of Adelaide, South Australia, 1982.

J. R. Arkinstall and P. R. Scott, An isoperimetric problem with lattice point constraints, *J. Austral. Math. Soc.* (*A*) **27** (1979), 27–36.

R. P. Bambah, A. Woods, and H. Zassenhaus, Three proofs of Minkowski's second inequality in the geometry of numbers, *J. Austral. Math. Soc.* **5** (1965), 453–462.

J. W. S. Cassels, On a theorem of Rado in the geometry of numbers, *J. London Math. Soc.* **22** (1947), 196–200; "An Introduction to the Geometry of Numbers," Springer-Verlag, Berlin/New York, 1959.

J. G. van der Corput, Verallgemeinerung einer Mordellschen Beweismethode in der Geometrie der Zahlen, *Acta Arith.* **1** (1935), 62–66; **2** (1936), 145–146.

J. G. van der Corput and H. Davenport, On Minkowski's fundamental theorem in the geometry of numbers, *K. Ned. Akad. vom Wetenschepper Proc.* **49/7** (1946), 701–707.

H. T. Croft, Cushions, cigars and diamonds—an area perimeter problem for symmetric ovals, *Math. Proc. Cambridge Philos. Soc.* **85** (1979), 1–16.

I. Danicic, An elementary proof of Minkowski's second inequality, *J. Austral. Math. Soc.* **10** (1969), 177–18.

H. Davenport, Minkowski's inequality for the minima associated with a convex body, *Quart. J. Math.* (*Oxford*) **10** (1939), 119–121.

E. Ehrhart, Une généralisation du théorème de Minkowski, *C. R. Acad. Sci.* (*Paris*) **240** (1955a), 483–485; Sur les ovales et les ovoides, *C. R. Acad. Sci.* (*Paris*) **240** (1955b), 583–585; Sur les ovales en géometrie des nombres, *C. R. Acad. Sci.* (*Paris*) **240** (1955c), 935–938; Propriétés arithmo-géometriques des ovales, *C. R. Acad. Sci.* (*Paris*) **241** (1955d), 274–276; Une généralisation probable du théorème fondamental de Minkowski, *C. R. Acad. Sci.* (*Paris*) **258** (1964), 4885–4887; Volume réticulaire critique d'un simplexe, *J. Reine Angew. Math.* **305** (1979), 218–220.

T. Estermann, Note on a theorem of Minkowski, *J. London Math. Soc.* **21** (1946), 179–182.

J. HAMMER, Some relatives of Minkowski's theorem for two-dimensional lattices, *Amer. Math. Monthly* **73** (1966), 744–746.

V. JARNIK, On the main theorem of the Minkowski geometry of numbers, *Časopis Pěst. Mat. Fys.* **73** (1948), 1–8; On Estermann's proof of a theorem of Minkowski, *Časopis Pěst. Mat. Fys.* **73** (1949), 131–140.

C. G. LEKKERKERKER, "Geometry of Numbers," Wolters-Noordhoff, Groningen, 1969.

K. MAHLER, Ein Übertragungsprinzip für konvexe Körper, *Časopis Pěst. Mat. Fys.* **68** (1939), 93–102.

Z. A. MELZAK, Minkowski's theorem with curvature limitations (I), *Cand. Math. Bull.* **2** (1959), 151–158.

H. MINKOWSKI, "Geometrie der Zahlen," Teubner, Leipzig, 1911.

L. J. MORDELL, On some arithmetical results in the geometry of numbers, *Compositio Math.* **1** (1934), 248–253.

R. RADO, A theorem on the geometry of numbers, *J. London Math. Soc.* **21** (1946), 34–47.

S. REICH, Two-dimensional lattices and convex domains, *Math. Mag.* **43** (1970), 219–220.

C. A. ROGERS, The product of the minima and determinant of a set, *Proc. Kon. Ned. Akad.* **52** (1949), 256–263.

D. B. SAWYER, The lattice determinants of asymmetrical convex regions, *J. London Math. Soc.* **29** (1954), 251–254; The lattice determinants of asymmetrical convex regions II, *Proc. London Math. Soc.* **3** (1955a), 197–217; The lattice determinants of asymmetrical convex regions III, *Quart. J. Math.* (*Oxford*) (2) **6** (1955b), 27–33.

P. R. SCOTT, On Minkowski's theorem, *Math. Mag.* **47** (1974a), 277; An analogue of Minkowski's theorem in the plane, *J. London Math. Soc.* **8** (1974b), 647–651; An area perimeter problem, *Amer. Math. Monthly* **81** (1974c), 884–885; Convex bodies and lattice points, *Math. Mag.* **48** (1975), 110–112; A new extension of Minkowski's theorem, *Bull. Austral. Math. Soc.* **18** (1978), 403–405; Two problems in the plane, *Amer. Math. Monthly* **89** (1982), 460–461; On planar convex sets containing one lattice point, *Quart. J. Math.* (*Oxford*) **36** (1985), 105–111.

H. P. F. SWINNERTON-DYER, Extremal lattices of convex bodies, *Proc. Cambridge Philos. Soc.* **49** (1953), 161–162.

B. UHRIN, Some useful estimations in the geometry of numbers, *Period. Math. Hungar.* **11** (1980), 95–103; On a generalization of Minkowski's convex body theorem, *J. Number Theory* **13** (1981), 192–209.

H. WEYL, On geometry of numbers, *Proc. London Math. Soc.* **47** (1942), 268–289.

G. K. WHITE, A refinement of van der Corput's theorem on convex bodies, *Amer. J. Math.* **85** (1963), 320–326.

J. M. WILLS, On an analogue to Minkowski's lattice point theorem, *in* "The Geometric Vein," pp. 285–288 (1982).

J. M. WILLS, J. ZAKS, AND M. A. PERLES, On lattice polytopes having interior lattice points, *Elem. Math.* **37/2** (1982), 44–46.

A. C. WOODS, On a theorem of Minkowski, *Proc. Amer. Math. Soc.* **9** (1958), 354–355.